Quantum error correction with the toric-GKP code

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Abstract
We examine the performance of the single-mode Gottesman-Kitaev-Preskill (GKP) code and its concatenation with the toric code for a noise model of Gaussian displacement errors.

Quantum oscillators and stabilizer codes
- Consider \( n \) quantum oscillators with conjugated observable: \( [\hat{p}_j, \hat{q}_k] = i\delta_{jk} \).
- Shift operators are given by
  \[
  \hat{U}(e) \equiv \prod_{k=1}^n e^{i\hat{p}_k+\delta_k}, \quad e \equiv (u, v) \in \mathbb{R}^2.
  \]
- With phases, they form a representation of the Heisenberg group, \( H_{su} \).
  \[
  \hat{U}(e) \hat{U}(e') = \hat{U}(e + e') \exp(i\omega), \quad \omega = u \cdot v' - v \cdot u'.
  \]
- Define the code-space as the common +1 eigenspace of an abelian subgroup, \( S \), of \( H_{su} \) called stabilizer group.
- Error model: Shifts with Gaussian amplitudes
  \[
  P(e) = \left( \frac{1}{\sqrt{2\pi}\sigma^2} \right)^{2n} \exp \left( \frac{e^2}{2\sigma^2} \right).
  \]

Maximal likelihood & Minimum Energy Decoders
- The probability of a given class of error \( [e + c] \) is given by
  \[
  P([e + c]) = \int_{s+c} P(s+c) P(e + s + c) \equiv Z_c(e).
  \]
- The ML and ME decoders correspond to:
  - ML: find \( \max Z_c(e) \)
  - ME: find \( \max P(e + s + c) \)
- Logical probability of error after ML decoding
  \[
  P_{ML}(c) = \int d\phi P(e) Z_{\text{e+ML}}(e) / \sum_{b} Z_{\text{e+ML}}(b).
  \]

Single GKP error correction
- The single mode GKP code is stabilized by
  \[
  S_p = e^{i\sqrt{\pi} \hat{q}} \quad \text{and} \quad S_y = e^{i\sqrt{\pi} \hat{p}}.
  \]
- Logical operators are \( \hat{X} = e^{i\sqrt{\pi} S_y} \) and \( \hat{Z} = e^{i\sqrt{\pi} S_p} \).
- Consider \( \Lambda \) rounds of noisy (X) error correction:
  \[
  q_j = \phi_j + \delta_j \mod 2\pi, \quad \phi_j - \phi_{j-1} = \epsilon_j, \quad \phi_0 \equiv 0.
  \]

Euclidean path integral for decoding
- Consider first perfect GKP error correction:
  - A GKP syndrome, \( q_i \), gives extra information:
    \[
    P(1|q_i) \propto \exp \left( \frac{(-2\pi + q_i + 4\pi k)^2}{2\sigma^2} \right).
    \]
  - This defines weights for MWPM:
    \[
    w_{ij} = \log \left( \frac{1 - P(1|q_i)}{P(1|q_j)} \right).
    \]

Decoding using GKP information
- Derive the model equivalent to the decoding problem:
  \[
  Z_{\text{GKP}}(e, \delta, \xi) \propto \int \mathcal{D}[\hat{A}] \exp(-H(e, \delta, \xi)[\hat{A}]).
  \]
- The full model:
  \[
  H(e, \delta, \xi)[\hat{A}] = \frac{1}{\sigma^2} \sum_{b} \cos(\delta_b - 2A_b)
  \]
- Decomposed differently:
  \[
  A_b = \delta_b/2 + \pi s_b + a_b, \quad (s_b = (-1)^{a_b})
  \]
  \[
  H(e, \delta, \xi)[\hat{A}] = H_{\text{PGM}}(e, \delta, \xi) \xi
  \]
  \[
  \frac{1}{\sigma^2} \sum_{b} \cos(2a_b)
  \]

Statistical physics model
- Place the rounds of syndrome extraction on a lattice:
- Make a time evolution of error on the lattice:
  \[
  \hat{A}(t) = \hat{A}_0 + \xi(t) + \hat{A}_0 + \hat{A}_0 + \cdots
  \]

Single GKP performance
- Simulating and comparing different decoders:
  \[
  M:\text{compute } Z_c(e) \text{ via Gaussian integrals}.
  \]
  \[
  \text{ME: Minimize } H(e, \delta, \xi) \text{ over } \phi.
  \]
- Comparing decoders using or not GKP information:

Improved threshold using GKP information
- (Left) Candidate phase diagram without disorder (\( \sigma_T = \sigma \)). (Below) Performance of two approximate ME decoders.

Toric-GKP code performance
- (Left) Candidate phase diagram without disorder (\( \sigma_T = \sigma \)). (Below) Performance of two approximate ME decoders.

Footnotes: